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A solvable sector of AdS theory

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Abstract: Field theory in space-time with boundary has an interesting sub-sector, where propagator is *difference* of those with Neumann and Dirichlet boundary conditions. Such boundary-induced theory in the bulk is essentially holomorphic and is exactly solvable in the sense that all orders of perturbation theory can be summed up explicitly into effective non-local theory at the boundary. This provides a non-trivial realization of holography principle. In the particular example of scalar fields of dimensions $\Delta_\pm = (d \pm 1)/2$ in AdS_{d+1} the corresponding effective conformal theory has propagators $|\vec{p}\,|^{-1}$ and vertices $(|\vec{p}_1| + \cdots + |\vec{p}_n|)^{-s_n}$ of valence n in momentum representation, with $s_n = (n-2)\Delta_- - 1$. This extraordinary simplicity of certain amplitudes in AdS seems inspiring and can be helpful for analyzing corollaries of open-closed string duality for particular field-theory sub-sectors of string theory.

Keywords: Field Theories in Higher Dimensions, AdS-CFT and dS-CFT Correspondence.

Contents

1. Introduction

Quantum field theory in the space-time with boundaries has additional structures as compared to the theory when no boundaries are present [1 – 8]. One of the most important questions in this context is relation between the theories in the bulk and at the boundary, where the latter one describes the eigen-states (wave-functions) of the evolution operators of the former. This relation is believed to be especially important [9, 10] in topological and gravitational bulk theories with trivial hamiltonians. Further, it is expected to be promoted to an even more fundamental open-closed string duality [11].

In this paper we consider a small subject in the story of bulk-to-boundary correspondence, which – to the best of our knowledge — did not attract enough attention in the literature. It concerns a possibility to form a nearly topological theory in the bulk, by substituting the bulk propagators by *differences* of those with Neumann and Dirichlet boundary conditions at the boundary. Such subtraction eliminates the physical pole from the propagator and — if the boundary consists of the stable points of discrete Z_2 transformation – substitutes it by "unphysical" pole at Z_2-image of the physical one. This situation is modeled by the theory of complex scalar field χ in the upper semi-space R^{d+1}_+ and the boundary R^d located at $z_0 = 0$, with partition function

$$\mathcal{Z}\{J\} = \int D\chi(z) D\bar{\chi}(z) \exp\left\{ -\frac{1}{g^2}\int_{R^{d+1}_+} \left(\partial_\mu\chi\partial_\mu\bar{\chi} + \sum_n \frac{t_n}{n!}\, z_0^{s_n-1}\chi^n \right) d^{d+1}z + \right.$$

$$\left. + \frac{1}{g^2}\oint_{R^d} \Big(J_N(\vec{x})\,\mathrm{Re}\,\chi(\vec{x}) + J_D(\vec{x})\,\mathrm{Im}\,\partial_0\chi(\vec{x}) \Big) d\vec{x} \right\} \tag{1.1}$$

where functional integral is over the fields with mixed (Neumann-Dirichlet) boundary conditions

$$\left. \begin{aligned} \mathrm{Re}\,\partial_0\chi(\vec{x}, z_0) &= 0 \\ \mathrm{Im}\,\chi(\vec{x}, z_0) &= 0 \end{aligned} \right|_{z_0=0} \tag{1.2}$$

and $\vec{x}$ are coordinates on the d-dimensional boundary (while coordinates in the bulk are $z_0, \vec{z}$). The summation over μ is with respect to the flat euclidean metric. Be there no boundary, the theory would be trivial, since propagator converts χ into $\bar{\chi}$, while interaction terms are pure holomorphic: contain only $\chi(z)$, not $\bar{\chi}(z)$. Because of the boundary conditions the cancellation between the contributions of propagating Re $\chi(z) = \frac{1}{2}(\chi+\bar{\chi})(z)$ and Im $\chi(z) = \frac{1}{2i}(\chi-\bar{\chi})(z)$ is not complete and partition function appears to be non-trivial. Still, it is particularly simple, and — as we demonstrate below — can be straightforwardly rewritten as an effective field theory of a single real field $\varphi(\vec{x})$ at the boundary,

$$\mathcal{Z}\{J\} = \int D\varphi(\vec{x}) \exp\left\{ -\frac{1}{g^2}\oint_{R^d} \left(\frac{1}{2}\varphi\sqrt{\Box}\varphi + \sum_n \frac{t_n}{n!}\int_0^\infty \left(e^{-\alpha\sqrt{\Box}}\varphi\right)^n \alpha^{s_n-1} d\alpha + \right.\right.$$

$$\left.\left. + \left(J_N + J_D\sqrt{\Box}\right)\varphi \right) d\vec{x} \right\} \tag{1.3}$$

Here $\Box = -\partial_{\vec{x}}\partial_{\vec{x}}$ is *minus* laplacian at the boundary and α is *not* a field, just a single auxiliary integrational variable.

The field $\bar{\chi}$ enters linearly in the action (1.1) and can work as Lagrange multiplier, providing a functional delta-function $\delta((\partial_0^2 - \Box)\chi(z) + J(\vec{x})\delta(z_0))$. However, in the presence of boundary this condition does not fix $\chi(z)$ unambiguously in the bulk: a *functional* freedom remains in the zero-modes of the Laplace operator in R^{d+1}_+ and it is not eliminated by our mixed boundary conditions. Thus the theory remains non-trivial, just its degrees of freedom are actually those of a field on the boundary – providing a non-trivial realization of holography idea.

This example can be easily extended to more general space-times with boundaries and boundaries can consist of the stable points of other Z_2 transformations.[1] For euclidean space R^{d+1} *per se* the index $s_n = 1$, while for

$$s_n = (n-2)\Delta_- - 1$$

[1]In (1.1) it is the symmetry $z_0 \to -z_0$, which in the case of $d+1 = 2$ with complex $z = z_1 + iz_0$ becomes complex conjugation $z \to \bar{z}$. Of considerable interest in the same dimension is the case when the

(1.1) and (1.3) describe a sub-sector in the theory of scalar fields $\phi_\pm$ of particular dimensions $\Delta_\pm = \frac{d\pm 1}{2}$ in AdS_{d+1}, with $\phi_{\Delta_-} = z_0^{\Delta_-}\,\mathrm{Re}\,\chi(z)$ and $\phi_{\Delta_+} = z_0^{\Delta_-}\,\mathrm{Im}\,\chi(z)$. Among other things, this fact manifests itself in *conformal invariance* of effective theory (1.3). Amusingly, even for the AdS_{d+1} case, s_n can sometime take value 1: this happens when interaction shape is adjusted to space-time dimension so that $(n-2)((d+1)-2) = 4$, i.e. when $(d+1, n) = (6,3)$, $(4,4)$ or $(3,6)$. To avoid possible confusion, we emphasize that the action (1.1) is essentially complex, still most amplitudes are real, but the unitarity of the theory (inessential for our considerations) can be under question.

The model (1.1), or equivalently AdS theory, is solvable in the sense that it can be explicitly projected onto the boundary, giving the theory (1.3), which is still dynamically non-trivial.

In the remaining part of this paper we briefly comment on straightforward derivations of (1.1) from the scalar theory in euclidean AdS_{d+1} and of (1.3) from (1.1).

2. Propagators in AdS

2.1 Action

The free action of real scalar field ϕ_Δ in AdS_{d+1} with euclidean metric

$$ds^2 = \frac{dz_0^2 + d\vec{z}^2}{z_0^2}$$

is given by

$$\begin{aligned} S &= \frac{1}{2}\int_{AdS_{d+1}} \sqrt{g}\left(g^{\mu\nu}\partial_\mu\phi_\Delta\partial_\nu\phi_\Delta + m_\Delta^2\phi_\Delta^2\right) d^{d+1}z + S_{\text{bound}} \\ &= \frac{1}{2}\int_{AdS_{d+1}} \frac{d^{d+1}z}{z_0^{d-1}}\left((\partial_0\phi_\Delta)^2 + (\vec{\partial}\phi_\Delta)^2 + \frac{m_\Delta^2\phi_\Delta^2}{z_0^2}\right) + S_{\text{bound}}\,, \end{aligned} \tag{2.1}$$

where the AdS mass is related to AdS dimension Δ of ϕ_Δ through

$$m_\Delta^2 = \Delta(\Delta - d)\,.$$

Dimension Δ is restricted to $\Delta > \frac{d-2}{2}$ by the unitarity bound [5, 7]. This action can be transformed to the one in R_+^{d+1} by rescaling of field variables[2] $\phi_\Delta(z) = z_0^{\Delta_-}\chi_\Delta(z)$:

$$S = \frac{1}{2}\int_{R_+^{d+1}} d^{d+1}z\left((\partial_0\chi_\Delta)^2 + (\vec{\partial}\chi_\Delta)^2 + \frac{m_\Delta^2 + \Delta_+\Delta_-}{z_0^2}\chi_\Delta^2\right), \tag{2.2}$$

role of space-time is played by a *hyperelliptic* Riemann surface $y^2(z) =$ Polynomial of z, while the discrete transformation is different: $y(z) \to -y(z)$. In this example our "unphysical", Neumann minus Dirichlet, propagator is exactly the one that appears in the role of the two-point function $\rho^{(0|2)}(z,z')$ in matrix model theory, with pole not at coincident, but at the Z_2-reflected points $z' = z^*$, lying at two different sheets of the surface, see [12, 13] for recent presentations. Actually, one can use in the theory (1.1) the ordinary, *physical*, propagator, but than the interaction term becomes explicitly non-local: made from powers of $\frac{1}{2}(\chi(z) + \bar{\chi}(z^*))$ instead of $\chi(z)$.

[2]We thank V.Rubakov for useful discussions of this issue.

provided in (2.1)

$$S_{\rm bound} = \frac{\Delta_-}{2}\oint_{\partial\left(AdS_{d+1}\right)} \frac{d^d\vec{z}}{z_0^d}\,\phi^2_\Delta = \frac{\Delta_-}{2}\oint_{R^d} d^d\vec{x}\,\lim_{z_0\to+0}\frac{\chi^2_\Delta(z_0,\vec{x})}{z_0}\,.$$

For special dimensions $\Delta = \Delta_\pm$ the AdS masses $m^2_{\Delta_\pm} = -\Delta_+\Delta_-$, so that the last term at the r.h.s. of (2.2) vanishes,[3] and the action converts into a free massless action for the scalar field in R^{d+1}:

$$S = \frac{1}{2}\int_{R_+^{d+1}} d^{d+1}z\left((\partial_0\chi_{\Delta_\pm})^2 + (\vec{\partial}\chi_{\Delta_\pm})^2\right). \tag{2.3}$$

One more peculiar feature of this particular choice of Δ is that precisely at $\Delta = \Delta_\pm$ the scalar theory in the bulk acquires extended symmetry: *local* conformal invariance. Indeed, since the scalar curvature $R(z)$ of *conformal* metric $g_{\mu\nu}(z) = \rho(z)\delta_{\mu\nu}$ in $d+1$ dimensions is equal to

$$R = -\frac{d}{\rho}\left(\partial^2\log\rho\; + \frac{d-1}{4}\partial_\mu\log\rho\;\partial_\mu\log\rho\right),$$

it follows that

$$\frac{1}{2}\int\sqrt{g}\left(g^{\mu\nu}\partial_\mu\phi_{\Delta_\pm}\partial_\nu\phi_{\Delta_\pm} + \xi R\phi^2_{\Delta_\pm}\right)d^{d+1}z$$

is invariant under the simultaneous change

$$\begin{cases} \rho(z) \longrightarrow \lambda^2(z)\rho(z)\,, \\ \phi_{\Delta_\pm}(z) \longrightarrow \lambda^{-\Delta_-}(z)\phi_{\Delta_\pm}(z) \end{cases}$$

with arbitrary z-dependent $\lambda(z)$, satisfying $\partial_0\lambda(z_0=0)=0$ provided

$$\xi = \frac{d-1}{4d} = \frac{\Delta_-}{2d}$$

(for $d+1 = 4$ this $\xi = \frac{1}{6}$). For AdS_{d+1} our $\rho = \frac{1}{z_0^2}$ and $R = -d(d+1)$, so that $\xi R(z) = -\frac{(d-1)(d+1)}{4} = -\Delta_-\Delta_+$, i.e. exactly equals $m^2_{\Delta_\pm}$. Interactions preserve conformal symmetry, both local and global, whenever $s_n = 1$. An interesting question is how the *local* symmetry is realized in these cases in the boundary theory (1.3).

2.2 Bulk-to-boundary propagators

Whenever the boundary is euclidean space — as is the case with our parametrization of AdS — it is practical to use *mixed* representation for Feynman diagrams: *momentum* along the boundary and *coordinate* in the orthogonal direction (inside the bulk). Fourier transform of the boundary variable $\vec{x}$ converts the AdS bulk-to-boundary propagator [4]

$$\tilde{K}_\Delta(w,\vec{x}) = \frac{w_0^\Delta}{\left[w_0^2 + (\vec{w}-\vec{x})^2\right]^\Delta} \tag{2.4}$$

[3]Potentially solvable (rational Calogero) situation arises for entire integer-labeled tower of "dressed" masses: $M^2_\Delta = \Delta(\Delta-d) + \Delta_+\Delta_- = N(N+1)$, i.e. for $\Delta = \frac{1}{2}\left(d \pm (2N+1)\right)$. We do not discuss this — as many other — obvious generalizations here.

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into[4]

$$K_\Delta(w|\vec{p}) = \int d^d\vec{x} e^{i\vec{p}\vec{x}} \frac{w_0^\Delta}{\left[w_0^2 + (\vec{w}-\vec{x})^2\right]^\Delta} = w_0^\Delta e^{i\vec{p}\vec{w}} \left(\frac{\vec{p}^2}{w_0^2}\right)^{\frac{1}{2}(\Delta-\frac{d}{2})} \mathcal{K}_{\Delta-\frac{d}{2}}\left(w_0\sqrt{\vec{p}^2}\right). \quad (2.5)$$

Here and below we systematically omit inessential factors to avoid overloading the formulas. Equalities are defined modulo such factors.

For semi-integer index $\Delta - \frac{d}{2}$ the modified Bessel function $\mathcal{K}$ turns into elementary function, especially simple for particular values of $\Delta = \Delta_\pm = \frac{1}{2}(d \pm 1)$. This follows from the intermediate formulas:

$$\begin{aligned} K_\Delta(w|\vec{p}) &= w_0^\Delta \int d^d\vec{x} e^{i\vec{p}\vec{x}} \int_0^\infty e^{-\alpha(w_0^2+(\vec{w}-\vec{x})^2)} \alpha^{\Delta-1} d\alpha \\ &= w_0^\Delta e^{i\vec{p}\vec{w}} \int_0^\infty e^{-\alpha w_0^2 - \frac{|p|^2}{4\alpha}} \alpha^{\Delta-\frac{d}{2}-1} d\alpha \\ &= w_0^\Delta e^{i\vec{p}\vec{w}} \int_0^\infty e^{-\beta^2 w_0^2 - \frac{|p|^2}{4\beta^2}} \beta^{2\Delta-d-1} d\beta \\ &= w_0^\Delta e^{i\vec{p}\vec{w}} \int_0^\infty e^{-\frac{1}{4}\lambda^2|p|^2 - \frac{w_0^2}{\lambda^2}} \lambda^{d-2\Delta-1} d\lambda\,. \end{aligned} \quad (2.6)$$

Here $\alpha = \beta^2 = \lambda^{-2}$ and $|p| = \sqrt{(\vec{p})^2}$. Distinguished are values of Δ when the powers of either β or λ disappear from the pre-exponents, i.e. when $\Delta = \Delta_\pm$. The integral over β (or λ) produces a factor of w_0^{-1} (or $|p|^{-1}$) in the pre-exponent,[5] and we get:

$$K_{\Delta_+}(w|\vec{p}) = w_0^{\Delta_-} e^{i\vec{p}\vec{w}} e^{-|p|w_0} \quad (2.7)$$

(since $\Delta_+ - 1 = \Delta_-$), and

$$K_{\Delta_-}(w|\vec{p}) = \frac{1}{|p|} w_0^{\Delta_-} e^{i\vec{p}\vec{w}} e^{-|p|w_0} = \frac{1}{|p|} K_{\Delta_+}(w|\vec{p})\,. \quad (2.8)$$

Eq. (2.8) implies that the amplitudes with external fields of dimension $\Delta = \Delta_-$ at the boundary can be obtained from those of $\Delta = \Delta_+$ by insertion of appropriate $|p|^{-1}$ factors.

[4]Schwinger parametrization is used to deal with the denominator:

$$\frac{1}{P^\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty d\lambda \lambda^{\alpha-1} e^{-\lambda P}$$

and the emerging integral is

$$\int_0^\infty \lambda^{\nu-1} e^{-\alpha\lambda - \frac{\beta}{\lambda}} d\lambda = 2\left(\frac{\beta}{\alpha}\right)^{\frac{\nu}{2}} \mathcal{K}_\nu\left(2\sqrt{\alpha\beta}\right).$$

[5]Here the celebrated integral is used:

$$\int_0^\infty e^{-a\lambda^2 - \frac{b}{\lambda^2}} d\lambda = \sqrt{\frac{\pi}{4a}}\, e^{-2\sqrt{ab}}$$

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2.3 Bulk-to-bulk propagators

Propagators in the space-time with boundary depend on boundary conditions. Consider two particular choices: Neumann and Dirichlet boundary conditions imposed at $z_0 = 0$. Thus, the propagator of the scalar field ϕ_Δ in AdS_{d+1} satisfies

$$\Big(- \Box_{AdS}(w) + m_\Delta^2 \Big) G_\Delta(w,z) = \frac{1}{\sqrt{g(z)}} \delta^{(d+1)}(w-z) = (z_0)^{d+1} \delta^{(d+1)}(w-z) \qquad (2.9)$$

with additional requirement that

$$G_\Delta(w,z) = w_0^\Delta \qquad \text{as} \qquad w_0 \to +0\,. \qquad (2.10)$$

Our consideration in subsection 2.1 implies that for particular values of $\Delta = \Delta_\pm$ this propagator is — modulo the factor of $(w_0 z_0)^{\Delta_-}$ — the massless scalar propagator in R^{d+1}, i.e. Fourier transform of $(p_0^2 + \vec{p}^{\,2})^{-1}$. The difference between Δ_+ and Δ_- is in the boundary condition (2.10):

$$\begin{aligned} G_{\Delta_+}(w,z) &= (w_0 z_0)^{\Delta_-} \int \frac{d\vec{p}\, dp_0}{p_0^2 + \vec{p}^{\,2}} e^{i\vec{p}(\vec{w}-\vec{z})} \sin(p_0 z_0) \sin(p_0 w_0) \\ &= (w_0 z_0)^{\Delta_-} \left\{ \frac{1}{((w_0 - z_0)^2 + (\vec{w}-\vec{z})^2)^{\Delta_-}} - \frac{1}{((w_0 + z_0)^2 + (\vec{w}-\vec{z})^2)^{\Delta_-}} \right\} \\ &= \frac{1}{u^{\Delta_-}} - \frac{1}{(u+2)^{\Delta_-}} \end{aligned} \qquad (2.11)$$

satisfies Dirichlet, while

$$\begin{aligned} G_{\Delta_-}(w,z) &= (w_0 z_0)^{\Delta_-} \int \frac{d\vec{p} dp_0}{p_0^2 + \vec{p}^{\,2}} e^{i\vec{p}(\vec{w}-\vec{z})} \cos(p_0 z_0) \cos(p_0 w_0) \\ &= (w_0 z_0)^{\Delta_-} \left\{ \frac{1}{((w_0 - z_0)^2 + (\vec{w}-\vec{z})^2)^{\Delta_-}} + \frac{1}{((w_0 + z_0)^2 + (\vec{w}-\vec{z})^2)^{\Delta_-}} \right\} \\ &= \frac{1}{u^{\Delta_-}} + \frac{1}{(u+2)^{\Delta_-}} \end{aligned} \qquad (2.12)$$

– Neumann boundary conditions at the boundary of AdS (at $w_0 = 0$ or $z_0 = 0$). Integrals in (2.11) and (2.12) are evaluated by introduction of auxiliary integration,

$$\frac{1}{p_0^2 + \vec{p}^{\,2}} = \int_0^\infty e^{-\alpha(p_0^2 + \vec{p}^{\,2})} d\alpha,$$

followed by gaussian integration over p_0 and $\vec{p}$. Switching to $\lambda = (4\alpha)^{-1}$ we obtain:

$$\begin{aligned} G_{\Delta_+}(w,z) = (w_0 z_0)^{\Delta_-} \int_0^\infty \lambda^{\Delta_- - 1} d\lambda \Big(&\exp\Big[-\lambda \Big((w_0 - z_0)^2 + (\vec{w}-\vec{z})^2 \Big) \Big] - \\ &- \exp\Big[-\lambda \Big((w_0 + z_0)^2 + (\vec{w}-\vec{z})^2 \Big) \Big] \Big) \end{aligned}$$

and integral over λ provides (2.11). The last representations in formulas (2.11) and (2.12) are in terms of the usual AdS variables

$$u = \frac{(w_0 - z_0)^2 + (\vec{w}-\vec{z})^2}{2 w_0 z_0} \qquad \text{and} \qquad u + 2 = \frac{(w_0 + z_0)^2 + (\vec{w}-\vec{z})^2}{2 w_0 z_0}\,.$$

For the sake of completeness, in appendix A at the end of the paper, we present alternative derivation of these propagators: as solutions of the hypergeometric equation.

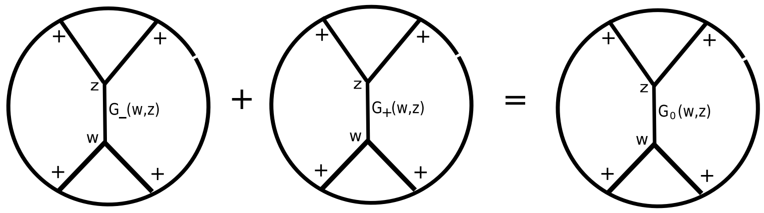

Figure 1: Four-point function. The contribution of $G_{\Delta_-}(w,z)$ and $G_{\Delta_+}(w,z)$ in the bulk gives propagator $G_0(w,z)$.

3. The difference of AdS propagators and the theory (1.1)

Now we are ready to introduce our simplified AdS theory. Suggestion is to substitute the propagators (2.12) and (2.11) by their peculiar linear combination: subtract one from another and define

$$G_0(w,z) \equiv \frac{1}{2}\Big(G_{\Delta_-}(w,z) - G_{\Delta_+}(w,z)\Big) = (w_0 z_0)^{\Delta_-} \int \frac{d\vec{p}dp_0}{p_0^2+\vec{p}^2} e^{i\vec{p}(\vec{w}-\vec{z})} \cos\Big(p_0(w_0+z_0)\Big)$$
$$= \frac{1}{(u+2)^{\Delta_-}} = \left(\frac{w_0 z_0}{(w_0+z_0)^2+(\vec{w}-\vec{z})^2}\right)^{\Delta_-}$$

as the propagator of a new conformally invariant scalar field theory. The spatial Fourier transform of such bulk-to-bulk propagator is especially simple:

$$G_0(w,z) = (w_0 z_0)^{\Delta_-} \int \frac{d^d\vec{p}}{|p|} e^{i\vec{p}(\vec{w}-\vec{z})} e^{-|p|(w_0+z_0)}\,. \tag{3.1}$$

The fact that z_0 and w_0 appear in the exponent as a simple sum,[6] makes convolutions of propagators in expressions for Feynman diagrams very simple and leads to especial simplicity of effective theory (1.3).

What is the way to realize such projection — from two different propagators to their difference? A possible answer is provided by the theory of two real scalar fields ϕ_{Δ_-} and ϕ_{Δ_+}, subjected to Neumann and Dirichlet boundary conditions respectively with the action

$$S = \int d^{d+1}z\sqrt{g}\bigg[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi_{\Delta_-}\partial_\nu\phi_{\Delta_-} + \frac{1}{2}m^2_{\Delta_\pm}\phi^2_{\Delta_-} + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi_{\Delta_+}\partial_\nu\phi_{\Delta_+} + \frac{1}{2}m^2_{\Delta_\pm}\phi^2_{\Delta_+} +$$
$$+\sum_n \frac{t_n}{n!}(\phi_{\Delta_-}+i\phi_{\Delta_+})^n\bigg]. \tag{3.2}$$

To prove that this theory reproduces the bulk-to-bulk propagator $G_0(w,z)$, consider the simplest case, when only $t_3 \neq 0$ in the interaction terms. In the figure 1, the tree contribution to the four-point function of ϕ_{Δ_+} is shown.

There are two diagrams: one with the bulk-to-bulk propagator $G_{\Delta_-}(w,z)$, another one with $G_{\Delta_+}(w,z)$. The vertex contribution for $\phi^3_{\Delta_+}$ interaction term is $V_{+++} = i^3 t_3\sqrt{g}$, while

[6]Unlike they show up in Fourier transform of the *physical* propagator $\mathcal{G} = \frac{1}{2}(G_{\Delta_-}+G_{\Delta_+}) \sim 1/u^{\Delta_-}$, which contains a far more sophisticated factor of $e^{i|p||w_0-z_0|}$, see appendix C below.

for $\phi^2_{\Delta_+}\phi_{\Delta_-}$ is $V_{++-} = i^2 t_3\sqrt{g}$. Thus, $V_{+++} = iV_{++-}$. This extra i factor, being squared (since there are two vertices in each diagram), gives relative *minus* sign between these two diagrams. Hence, effectively, the propagator in this theory is $G_{\Delta_-}(w,z) - G_{\Delta_+}(w,z) = G_0(w,z)$. The generalization to other tree and loop diagrams is straighforward.

In terms of fields

$$\begin{aligned} \phi &= \phi_{\Delta_-} + i\phi_{\Delta_+} \\ \overline{\phi} &= \phi_{\Delta_-} - i\phi_{\Delta_+} \end{aligned}$$

this action can be rewritten as

$$S = \int d^{d+1}z\sqrt{g}\left(\frac{1}{2}g^{\mu\nu}\,\partial_\mu\phi\partial_\nu\overline{\phi} + \frac{m^2_{\Delta_\pm}}{2}\phi\overline{\phi} + \sum_n \frac{t_n}{n!}\phi^n\right). \tag{3.3}$$

Making rescaling of variables

$$\begin{aligned} \phi_{\Delta_-} &= z_0^{\Delta_-}\chi_{\Delta_-} \\ \phi_{\Delta_+} &= z_0^{\Delta_-}\chi_{\Delta_+} \end{aligned}$$

from the section 2.1, we obtain, up to boundary terms, the action

$$S = \int d^{d+1}z\left[\frac{1}{2}\partial_\mu\chi\partial_\mu\overline{\chi} + \sum_n \frac{t_n}{n!}z_0^{s_n-1}\chi^n\right] \tag{3.4}$$

for the fields $\chi = \chi_{\Delta_-} + i\chi_{\Delta_+}$ and $\overline{\chi} = \chi_{\Delta_-} - i\chi_{\Delta_+}$. Index $s_n = (n-2)\Delta_- - 1$.

4. Tree diagrams

In this section we evaluate arbitrary tree correlators of the fields of dimension Δ_+ on the boundary of AdS_{d+1} in the complex-scalar field theory (1.1), or equivalently (3.3) with *holomorphic* interaction. It turns out that all the tree diagrams are expressed as simple rational functions of d-dimensional (rather than $d+1$-dimensional) momenta: one extra dimension can be explicitly integrated out and provide a non-local, but rather simple effective theory (1.3). Moreover, the same effective theory appears to describe all loop diagrams as well.

4.1 Single-vertex diagram

We begin from the simplest diagram with a single vertex of valence $n+1$, see figure 2.

The corresponding amplitude is given by

$$\begin{aligned} A(w|\vec{p}_1,\ldots,\vec{p}_n) \quad &= \quad t_{n+1}\int\frac{dz_0 d^d\vec{z}}{z_0^{d+1}}G_0(w,z)K_{\Delta_+}(z|\vec{p}_1)\ldots K_{\Delta_+}(z|\vec{p}_n) \\ &\overset{(3.1)\&(2.7)}{=} \quad t_{n+1}\int\frac{dz_0 d^d\vec{z}}{z_0^{d+1}}\left((w_0z_0)^{\Delta_-}\int\frac{d\vec{p}}{|p|}e^{i\vec{p}(\vec{w}-\vec{z})}e^{-|p|(w_0+z_0)}\right)\times \\ &\qquad \times\left(z_0^{\Delta_-}e^{i\vec{p}_1\vec{z}}e^{-|p_1|z_0}\right)\cdots\left(z_0^{\Delta_-}e^{i\vec{p}_n\vec{z}}e^{-|p_n|z_0}\right) \\ &= \quad t_{n+1}\frac{w_0^{\Delta_-}}{|p_{1n}|\Big(|p_1|+\cdots+|p_n|+|p_{1n}|\Big)^{s_{n+1}}}e^{i\vec{w}(\vec{p}_1+\cdots+\vec{p}_n)}e^{-|p_{1n}|w_0} \end{aligned} \tag{4.1}$$

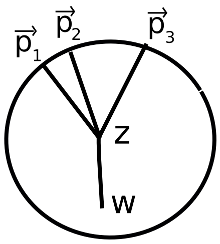

Figure 2: One-vertex diagram, w is arbitrary point in the bulk. Four-valent vertex is chosen for illustrative purposes.

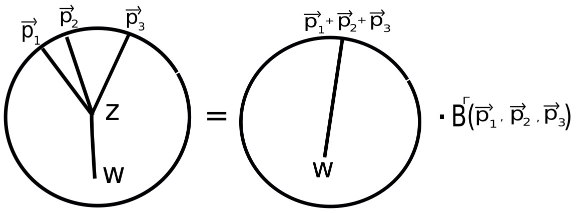

Figure 3: First step. Integration over $d^{d+1}z$ with the weight $G_0(w,z)$ converts the bunch of bulk-to-boundary propagators into a single bulk-to-boundary propagator.

where $|p_{1n}| \equiv \sqrt{(\vec{p}_1 + \cdots + \vec{p}_n)^2}$. Note, that the use of projection (3.1) instead of (2.11) or (2.12) eliminates the poles at collinear momenta, when at $|p_{1n}| = |p_1| + \cdots + |p_n|$, which normally occur in AdS Feynman diagrams, see appendix C below.

4.2 Multi-vertex diagrams

Comparison of (4.2) and (2.7) shows that their w-dependencies are exactly the same. This implies *universality* of the vertex insertion and the possibility of recursive evaluation of tree diagrams. Namely we can factor out all the w dependence in a simple and universal manner:

$$\begin{aligned} A^\Gamma(w|\vec{p}_1,\ldots,\vec{p}_n) &= w_0^{\Delta_-} e^{i\vec{w}(\vec{p}_1+\cdots+\vec{p}_n)} e^{-w_0|\vec{p}_1+\cdots+\vec{p}_n|} B^\Gamma(\vec{p}_1,\ldots,\vec{p}_n) \\ &= K_{\Delta_+}(w|\vec{p}_1+\cdots+\vec{p}_n) B^\Gamma(\vec{p}_1,\ldots,\vec{p}_n)\,, \end{aligned} \tag{4.2}$$

where Γ labels the graph = diagram, which in our case is the rooted tree with the total momentum $\vec{p}_\Gamma \equiv \vec{p}_1 + \cdots + \vec{p}_n$.

Eq. (4.2) is pictorially represented in figure 3. It shows that in the theory (1.1) or equivalently (3.3) any number of bulk-to-boundary propagators, attached to an intermediate vertex at z in the bulk, can after integration over z be substituted by a single bulk-to-boundary propagator, leading to the point w in the bulk and carrying the total along-the-boundary momentum, multiplied by the factor $B^\Gamma(\vec{p}_1,\ldots,\vec{p}_n)$.

At the next step we evaluate the second-level diagram like the left one in figure 4 with several single-vertex sub-diagrams meeting at the single vertex at point y in the bulk.

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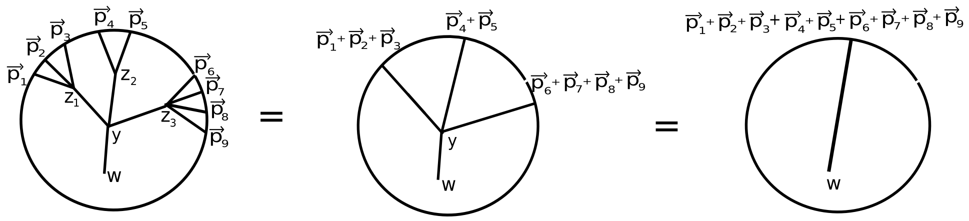

Figure 4: Second step. Factors $B(\vec{p}_i)$ omitted.

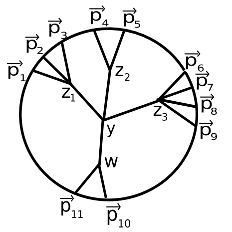

Figure 5: One loop diagram.

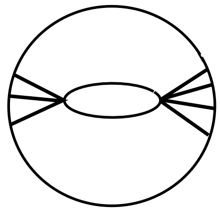

Figure 6: The last step. Emission of the bunch of bulk-to-boundary propagators from the final vertex at the point w in the bulk.

As shown in the picture, repeated application of above-described contraction provides an amplitude, again proportional to a single bulk-to-boundary propagator.

We can apply this algorithm as many times as necessary, converting arbitrary tree diagram with the root at w inside the bulk into a single bulk-to-boundary propagator $K_{\Delta_+}(w|\text{total momentum})$. The coefficient function arises from recurrent relation

$$B^{\Gamma} = t_{n+1} \frac{B^{\Gamma_1} \dots B^{\Gamma_n}}{|p_{\,\Gamma}| \Big(|p_{\,\Gamma_1}| + \dots + |p_{\,\Gamma_n}| + |p_{\,\Gamma}| \Big)^{s_{n+1}}} \,. \tag{4.3}$$

From this relation it is clear, that induced diagrammatic technique, is *local*: each vertex contribution depends on momenta, incoming into this particular vertex only. This *locality* is due to specific choice of propagator $G_0(w, z)$.

To finish evaluation of the diagram in figure 6 it remains to attach the last bunch of bulk-to-boundary propagators to the root w of the tree (figure 4 left) and integrate over w. Such convolution of m propagators is equal to:

$$\int K_{\Delta_+}(w|\vec{p}_1) \dots K_{\Delta_+}(w|\vec{p}_m) \frac{dw_0 d\vec{w}}{w_0^{d+1}} = \frac{\delta(\vec{p}_1 + \dots + \vec{p}_m)}{\Big(|p_1| + \dots + |p_m| \Big)^{s_m}} \,. \tag{4.4}$$

Following this procedure one can explicitly evaluate any particular tree diagram.

4.3 Feynman rules and effective action

The result of our consideration can be formulated as simple Feynman rules for the B^{Γ} functions (see table 1).

External lines carry the same factors $|p|^{-1}$ provided external fields have dimension Δ_-, while for dimension Δ_+ external lines carry no factors of momentum. Moreover, as explained in section 5 below, the same rules work for loop calculations.

propagator (line)	$\frac{1}{\lvert p\rvert}$
n-vertex	$t_n \frac{\delta^{(d)}(\vec{p}_1+\vec{p}_2+\cdots+\vec{p}_n)}{(\lvert p_1\rvert+\lvert p_2\rvert+\cdots+\lvert p_n\rvert)^{s_n}}$

Table 1: Feynman rules.

This diagram technique can be summarized in the form of a simple effective action: tree and loop diagrams for scalars of dimension $\Delta_\pm$ at the boundary and peculiar propagator $G_0(w,z)$ in the bulk coincide with the tree and loop diagrams in the *boundary* theory (1.3):

$$\begin{aligned}\mathcal{Z}\{J\} = \int D\varphi(\vec{x}) \exp\Big\{ -\frac{1}{g^2}\oint d\vec{x}\Big(\frac{1}{2}\varphi\sqrt{\Box}\varphi + \Big(J_N + J_D\sqrt{\Box}\Big)\varphi + \\ + \sum_n \frac{t_n}{n!}\int_0^\infty \Big(e^{-\alpha\sqrt{\Box}}\varphi\Big)^n \alpha^{s_n-1}d\alpha\Big)\Big\}. \qquad (4.5)\end{aligned}$$

This theory is conformal invariant: rescalings of field φ, which has dimension Δ_- are accompanied by the transformation of auxiliary integration variable α, which has dimension -1. In variance with the boundary models, usually considered in the context of AdS/CFT correspondence [3−6, 11, 14−17], the theory (4.5) is non-local. Also trees (loops) in the bulk are the same trees (loops) in the theory (4.5).

We emphasize that matching between theories (1.1) and (4.5) is valid for arbitrary value of index s_n (not restricted to $s_n = (n-2)\Delta_- - 1$). Only for particular choice of $s_n = (n-2)\Delta_- - 1$, the bulk theory (1.1) describes the AdS theory (3.3).

5. Loops

It is straightforward to see that our Feynman rules — and thus the effective theory (1.3) at the boundary — reproduce expressions for loop diagrams in AdS theory. Again, in variance with the usual AdS/CFT correspondence, loops in (3.3) are loops in (4.5).

5.1 Sample 1-loop diagram.

We provide just an illustration. Consider the diagram in figure 5.

Original expression is

$$\begin{aligned}t_n t_m \int\int \frac{dz_0 d^d\vec{z}}{z_0^{d+1}}\frac{dw_0 d^d\vec{w}}{w_0^{d+1}} K_{\Delta_+}(z|\vec{p}_1)\cdots\times \\ \times K_{\Delta_+}(z|\vec{p}_{n-2}) G_0^2(z,w) K_{\Delta_+}(w|\vec{q}_1)\cdots K_{\Delta_+}(w|\vec{q}_{m-2}) \,. \qquad (5.1)\end{aligned}$$

According to (2.7) and (3.1) the product of propagators is equal to

$$\begin{aligned}z_0^{\Delta_-(n-2)} e^{i\vec{z}\sum_i \vec{p}_i} e^{-z_0\sum_i |p_i|}\left((z_0 w_0)^{\Delta_-}\int \frac{d^d\vec{r}}{|r|} e^{i\vec{r}(\vec{z}-\vec{w})} e^{-|r|(w_0+z_0)}\right)^2 \times \\ \times w_0^{\Delta_-(m-2)} e^{i\vec{w}\sum_j \vec{q}_j} e^{-w_0\sum_j |q_j|}\end{aligned}$$

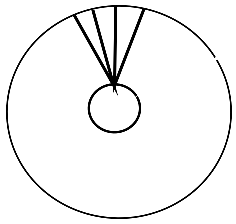

Figure 7: Tadpole diagram.

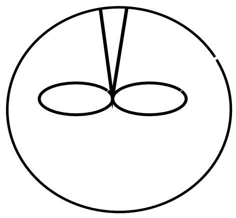

Figure 8: Two loop diagram, associated with the coupling t_6.

and after integration over $\vec{z}$ and $\vec{w}$ (5.1) turns into

$$\int\int \frac{d^d\vec{r}}{|r|}\frac{d^d\vec{r}'}{|r'|}\delta^{(d)}\left(\sum_i \vec{p}_i + \vec{r} + \vec{r}'\right)\delta^{(d)}\left(\sum_j \vec{q}_j - \vec{r} - \vec{r}'\right)\int_0^\infty z_0^{(n-2)\Delta_- -2}dz_0 \times$$

$$\times \int_0^\infty w_0^{(m-2)\Delta_- -2} dw_0 e^{-(\sum_i |p_i|+|r|+|r'|)z_0} e^{-(\sum_j |q_j|+|r|+|r'|)w_0} =$$

$$= t_n t_m \delta^{(d)}\left(\sum_i \vec{p}_i + \sum_j \vec{q}_j\right) \times$$

$$\times \int \frac{d^d\vec{r}}{|r||r'|\Big(\sum_i |p_i| + |r| + |r'|\Big)^{s_n}\Big(\sum_j |q_j| + |r| + |r'|\Big)^{s_m}}$$

where in the last formula $|r'| = \sqrt{(\sum_i \vec{p}_i + \vec{r})^2} = \sqrt{(\sum_j \vec{q}_j - \vec{r})^2}$. This expression is exactly the one which arises in effective theory (1.3) for the loop diagram of the same topology, figure 5.

Generalizations to all other loop diagrams is straightforward.

5.2 Tadpoles and divergencies

A few comments are deserved by tadpole diagrams, like those shown in figures 7, 8. First of all, the projected propagator $G_0(z,w)$ — in variance from the usual ones, like $G_{\Delta_\pm}(z,w)$ – is not singular at coincident points:

$$G_0(z,z) = z_0^{2\Delta_-}\int \frac{d^d\vec{p}}{|p\,|}e^{-2|p\,|z_0} = \text{const.}$$

(this is also clear from its expression through u-variable, since at $w = z$ this $u = 0$, but $u + 2 = 2 \neq 0$). Thus the UV divergences are absent, as one expects in the non-local theory (1.3) with exponential damping of interactions at large momenta.

However, some peculiar divergences still survive. For example, for the diagram in figure 7 we have, according to our Feynman rules:

$$t_n \delta^{(d)}\left(\sum_{i=1}^{n-2}\vec{p}_i\right)\int \frac{d^d\vec{q}}{|q|\Big(2|q| + \sum_{i=1}^{n-2}|p_i|\Big)^{s_n}} = t_n \frac{\delta^{(d)}\left(\sum_{i=1}^{n-2}\vec{p}_i\right)}{\Big(\sum_{i=1}^{n-2}|p_i|\Big)^{(n-4)\Delta_- -1}} \tag{5.2}$$

More accurately, expression in the r.h.s. is valid only when $(n-4)\Delta_- > 1$, otherwise the integral in the l.h.s. diverges at large $|q|$. (Note, that vanishing sum of the space momenta $\sum \vec{p}$ does not imply that the sum of their moduli, $\sum |p|$, is zero. This quantity is always positive.) If we rewrite (5.2) in coordinate representation, making use of $G_0(z,z) = const$, we get:

$$t_n \int \frac{dz_0 d^d\vec{z}}{z_0^{d+1}} K_{\Delta_+}(z|\vec{p}_1)\dots K_{\Delta_+}(z|\vec{p}_{n-2}) G_0(z,z) =$$
$$= t_n \delta^{(d)}\left(\sum_{i=1}^{n-2} \vec{p}_i\right) \int_0^\infty e^{-z_0 \sum_i |p_i|} z_0^{(n-4)\Delta_- -2} dz_0$$

In this representation it is clear that divergence, though looking like UV from the point of theory (1.3), comes from the region of small z_0, from the vicinity of the boundary of AdS and thus actually an IR divergence.

Particular diagram in figure 7 converges if $n \geq 6$, we consider $d \geq 2$, and divergence can be eliminated, say, by putting $t_3 = t_4 = t_5 = 0$. However, the two-loop diagram in figure 8, associated with the coupling t_6, diverges for exactly the same reason as the one loop in figure 7.

5.3 Coleman-Weinberg potential

The boundary theory (1.3) can be used to sum up the one-loop diagrams in the background of *constant* field Φ. The result of this summation — the Coleman-Weinberg action for our theory — is[7]

$$S_{CW}\{\Phi\} \sim \log \mathrm{Det}\left(\sqrt{\Box} + \sum_n \frac{t_n \Phi^{n-2}}{(n-2)!} \frac{1}{(2\sqrt{\Box})^{s_n}}\right) \longrightarrow$$
$$\longrightarrow \int \log\left(1 + \sum_n \frac{t_n}{(n-2)!} \frac{\Phi^{n-2}}{|q|(2|q|)^{s_n}}\right) d^d\vec{q}. \qquad (5.3)$$

For $s_n + 1 > d$, i.e. for $(n-4)\Delta_- > 1$ the integral converges at large $|q|$ (otherwise there is our familiar IR divergence at small z_0), and it converges for small $|q|$, though individual terms of power expansion in Φ are divergent.

It is instructive to compare this Coleman-Weinberg action with the naive open-sector boundary effective action, associated with the free-field theory with quadratic vertex operators:

$$Z_{op}\{I\} = \int D\tilde{\varphi}(\vec{x}) \exp\left\{-\frac{1}{g^2} \oint d^d\vec{x} \left(\tilde{\varphi}\Box\tilde{\varphi} + I\tilde{\varphi}^2\right)\right\} \sim \mathrm{Det}^{-1/2}(\Box + I). \qquad (5.4)$$

[7]We use the standard background field technique. The field φ is decomposed into the sum of background and quantum field: $\varphi = \Phi + \varphi_q$. We find the part of the action proportional to φ_q^2, then diagonalize it in momentum representation. Arrow implies the use of relation $\log \mathrm{Det} = \mathrm{Tr} \log$ and subtraction of the Φ-independent part from $S_{CW}(\Phi)$. The one-loop contribution to $\log \mathcal{Z}\{J\}$ is related to $S_{CW}(\Phi)$ by Legendre transform.

For constant source $I(\vec{x}) = const$

$$S_{op}(I) = \log Z_{op}\{I\} \sim \int \log\left(1 + \frac{I}{q^2}\right)\, d^d\vec{q} \tag{5.5}$$

and does not have anything to do with $\log \mathcal{Z}\{J\}$. Still, for distinguished value of index $s_n = 1$ and for $I = \Phi^{n-2}$, it coincides with $S_{CW}(\Phi)$ — the one-loop contribution to Legendre transform of $\log \mathcal{Z}\{J\}$ at constant Φ. Note also that the dimensions $\Delta = d - 2$, considered in [11] coincide with our $\Delta_\pm = \frac{d\pm1}{2}$ for $d = 3$ and $d = 5$.

A. Propagators as hypergeometric functions

Usually in the literature eq. (2.9) is not solved explicitly for particular dimensions $\Delta = \Delta_\pm$, as we did in this paper. Instead one uses the transcendental expression for the bulk-to-bulk propagator through hypergeometric series [14, 5, 11],

$$G_\Delta(w,z) = \frac{1}{(4\pi)^{\frac{d+1}{2}}} \frac{\Gamma(\Delta)\Gamma(\Delta-\Delta_-)}{\Gamma(2\Delta-2\Delta_-)} \left(\frac{2}{u}\right)^\Delta F\left(\Delta, \Delta-\Delta_-, 2\Delta-2\Delta_-; -\frac{2}{u}\right). \tag{A.1}$$

In this section we briefly explain, how our simple calculations are related to this standard approach.

For particular dimensions $\Delta = \Delta_\pm$, up to an overall normalization factor $1/(4\pi)^{\frac{d+1}{2}}$, we get[8] from (A.1):

$$\begin{aligned} G_{\Delta_-} &= \frac{2^{\Delta_-+1}\Gamma(\Delta_-)}{u^{\Delta_-}} F\left(\Delta_-, 0, 0; -\frac{2}{u}\right) \\ &= \frac{2^{\Delta_-}\Gamma(\Delta_-)}{u^{\Delta_-}} \left(1 + \sum_{n=0}^{\infty} \frac{\Gamma(n+\Delta_-)}{\Gamma(\Delta_-)n!} \left(-\frac{2}{u}\right)^n\right) \\ &= 2^{\Delta_-}\Gamma(\Delta_-)\left(\frac{1}{u^{\Delta_-}} + \frac{1}{(u+2)^{\Delta_-}}\right) \end{aligned} \tag{A.2}$$

and

$$G_{\Delta_+} = \frac{2^{\Delta_+}\Gamma(\Delta_+)}{u^{\Delta_+}} F\left(\Delta_+, 1, 2; -\frac{2}{u}\right) = \frac{2^{\Delta_+}}{u^{\Delta_+}} \sum_{n=0}^{\infty} \frac{\Gamma(n+\Delta_+)\Gamma(n+1)}{n!\Gamma(n+2)} \left(-\frac{2}{u}\right)^n$$

[8] Here we use the definition of Gauss hypergeometric series ${}_2F_1$,

$$F(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(n+a)\Gamma(n+b)}{n!\Gamma(n+c)} z^n\,,$$

which solves the hypergeometric equation,

$$z(1-z)F'' + \Big(c - (a+b+1)z\Big)F' - abF = 0\,,$$

and Newton's binomial expansion

$$(1-z)^{-s} = \sum_{n=0}^{\infty} \frac{\Gamma(n+s)}{n!\Gamma(s)} z^n = F(s,c,c;z)$$

$$
\begin{aligned}
&= -\frac{2^{\Delta_+}}{2u^{\Delta_-}}\sum_{n=1}^{\infty}\frac{\Gamma(n+\Delta_-)}{n!}\left(-\frac{2}{u}\right)^n \\
&= 2^{\Delta_-}\Gamma(\Delta_-)\left(\frac{1}{u^{\Delta_-}}-\frac{1}{(u+2)^{\Delta_-}}\right). \qquad \text{(A.3)}
\end{aligned}
$$

An extra coefficient 2 in the first line of (A.2) comes from the fact that the ratio $\frac{\Gamma(\Delta-\Delta_-)}{\Gamma(2\Delta-2\Delta_-)} \to 2$ as $\Delta \to \Delta_-$. For the same reason the $n=0$ term in the sum enters with an extra coefficient 2, this is taken into account by an extra item 1 in the first formula in the second line of (A.2).

Up to a common normalization factor the formulas (A.2) and (A.3) reproduce (2.12) and (2.11) respectively.

B. Derivations *à la* [14]

We illustrate this standard method by re-examining the example of (4.2), with $G_0(w,z)$ replaced by $G_\Delta(w,z)$. We put $n=3$ (triple vertex) and $d=5$ to simplify the formulas. The results of this appendix can be generalized to arbitrary values of n and d in an obvious way. Following [14] we extract this quantity from solution of the equation (2.9):

$$
(-\Box_{AdS}+m_\Delta^2)A_\Delta(w|\vec{p}_1,\vec{p}_2) = K_{\Delta_\pm}(w|\vec{p}_1)K_{\Delta_\pm}(w|\vec{p}_2) \sim e^{-(|p_1|+|p_2|)w_0}e^{i(\vec{p}_1+\vec{p}_2)\vec{w}}w_0^{2\Delta_-} \qquad \text{(B.1)}
$$

the subscript Δ for $A_\Delta(w|\vec{p}_1,\vec{p}_2)$ denotes dimension of the bulk-to-bulk propagator, eq. (B.2) below restricts it to be $\Delta=\Delta_\pm$. Proportionality sign in (B.1) shows that the possible $|p_1|^{-1}$ and $|p_2|^{-1}$ factors from K_{Δ_-} are ignored. Substituting $A_\Delta(w|\vec{p}_1,\vec{p}_2) = e^{i\vec{w}(\vec{p}_1+\vec{p}_2)}e^{-t\mu}F(t)$ with $t=w_0\sqrt{(\vec{p}_1+\vec{p}_2)^2}=w_0|p_{12}|$ and $\mu=\frac{|p_1|+|p_2|}{|p_{12}|}$, we obtain for $F(t)$:

$$
-t^2F''+(2\mu t^2+(d-1)t)F'+\big((1-\mu^2)t^2-\mu(d-1)t+\Delta(\Delta-d)\big)F = t^{2\Delta_-}|p_{12}|^{-2\Delta_-}
$$

Substituting further $F(t)=t^{\Delta_-}f(t)$, we get:

$$
-f''+2\mu f'+(1-\mu^2)f = t^{-2+\Delta_-}|p_{12}|^{-2\Delta_-} \qquad \text{(B.2)}
$$

Generic solution of this equation is:

$$
A_\Delta(w|\vec{p}_1,\vec{p}_2) = \frac{w_0^{\Delta_-}}{|p_{12}|^2-(|p_1|+|p_2|)^2}e^{i\vec{w}(\vec{p}_1+\vec{p}_2)}\left[e^{-w_0(|p_1|+|p_2|)}+c_1e^{-|p_{12}|w_0}+c_2e^{|p_{12}|w_0}\right] \qquad \text{(B.3)}
$$

Parameter c_2 should vanish, $c_2=0$, to prevent growth *inside* the bulk. Near the boundary, as $w_0\to 0$, $A_\Delta(w)\sim w_0^\Delta$. Therefore for $\Delta=\Delta_+=\Delta_-+1$ we need $c_1=-1$ so that the asymptotics of two terms in square brackets cancel each other at $w_0=0$. For $\Delta=\Delta_-$ asymptotic itself is correct for any $c_1\neq -1$, and is not enough to choose the right solution. However, as $w_0\to 0$ the $\vec{w}$-Fourier transform of A_{Δ_-} should be symmetric in all the three momenta, provided that $K_{\Delta_-}(w|\vec{p}_1)$ and $K_{\Delta_-}(w|\vec{p}_2)$ are chosen for the external legs:

$$
\begin{aligned}
S(\vec{p}_1,\vec{p}_2,\vec{p}_3) &\equiv \lim_{w_0\to 0} w_0^{-\Delta_-}\int A_{\Delta_-}(w_0,\vec{w}|\vec{p}_1,\vec{p}_2)e^{i\vec{w}\vec{p}_3}d\vec{w} \\
&= \frac{\delta^{(d)}(\vec{p}_1+\vec{p}_2+\vec{p}_3)}{|p_1||p_2||p_3|(|p_1|+|p_2|+|p_3|)} \times \frac{1+c_1}{|p_3|-|p_1|-|p_2|}
\end{aligned}
$$

$\delta^{(d)}$-function allows to substitute $|p_3| = \sqrt{\vec{p}_3^2}$ instead of $|p_{12}| = \sqrt{(\vec{p}_1+\vec{p}_2)^2}$. The last ratio breaks the symmetry unless $c_1 = -\frac{|p_1|+|p_2|}{|p_{12}|}H(\vec{p}_1,\vec{p}_2,\vec{p}_3)$. Here $H(\vec{p}_1,\vec{p}_2,\vec{p}_3)$ - is some symmetric function of its arguments. To find this function we need to fix the asymptotic of $A_\Delta(w|\vec{p}_1,\vec{p}_2) \sim Cw_0^\Delta$ at $w_0 \to 0$. This can be done by direct evaluation of integral in (4.2) and in appendix C for $w_0 = 0$. The result is $H(\vec{p}_1,\vec{p}_2,\vec{p}_3) = 1$. Thus, $c_1 = -\frac{|p_1|+|p_2|}{|p_{12}|}$. With such choice of c_1, for $\Delta = \Delta_\pm$, the pole at $|p_{12}| = |p_1| + |p_2|$, i.e. at collinear momenta $\vec{p}_1||\vec{p}_2$ disappears from the amplitude (but only at $w_0 = 0$).

C. Single-vertex diagram in *conventional* AdS theory

It is instructive to repeat the calculation from s. 4.1 for *conventional* AdS theory, with the same fields of dimensions $\Delta_\pm$, but with the *physical* bulk-to-bulk propagator

$$\mathcal{G}(w,z) = \left(\frac{w_0 z_0}{(w_0-z_0)^2 + (\vec{w}-\vec{z})^2}\right)^{\Delta_-} = (w_0 z_0)^{\Delta_-} \int \frac{d^d\vec{p}}{|p|} e^{i\vec{p}(\vec{w}-\vec{z})} e^{-|p||w_0-z_0|} \tag{C.1}$$

instead of our projected $G_0(w,z)$ from (3.1). The technical difference is that $w_0 - z_0$ can change sign and thus the integral over z_0 in the analogue of (4.2) is more sophisticated:

$$\begin{aligned}
\mathcal{A}(w|\vec{p}_1,\ldots,\vec{p}_n) \quad &= \quad \int \frac{dz_0 d^d\vec{z}}{z_0^{d+1}} \mathcal{G}(w,z) K_{\Delta_+}(z|\vec{p}_1)\ldots K_{\Delta_+}(z|\vec{p}_n) \\
&\overset{(C.1)\&(2.7)}{=} \int \frac{dz_0 d^d\vec{z}}{z_0^{d+1}} \left((w_0 z_0)^{\Delta_-} \int \frac{d\vec{p}}{|p\,|} e^{i\vec{p}(\vec{w}-\vec{z})} e^{-|p\,||w_0-z_0|}\right) \times \\
&\qquad \times \left(z_0^{\Delta_-} e^{i\vec{p}_1\vec{z}} e^{-|p_1|z_0}\right) \cdots \left(z_0^{\Delta_-} e^{i\vec{p}_n\vec{z}} e^{-|p_n|z_0}\right) \\
&= \quad \frac{w_0^{\Delta_-}}{|p_{1n}|} e^{i\vec{w}(\vec{p}_1+\cdots+\vec{p}_n)} \times \\
&\qquad \times \Bigg(e^{-|p_{1n}|w_0} \int_0^{w_0} e^{-(|p_1|+\cdots+|p_n|-|p_{1n}|)z_0} z_0^{s_{n+1}-1} dz_0 + \\
&\qquad\qquad + e^{|p_{1n}|w_0} \int_{w_0}^{\infty} e^{-(|p_1|+\cdots+|p_n|+|p_{1n}|)z_0} z_0^{s_{n+1}-1} dz_0\Bigg) \\
&= \quad \frac{w_0^{\Delta_-}}{|p_{1n}|} e^{i\vec{w}(\vec{p}_1+\cdots+\vec{p}_n)} \times \\
&\qquad \times \Bigg\{ \frac{\Gamma(s_{n+1})}{(|p_1|+\cdots+|p_n|-|p_{1n}|)^{s_{n+1}}} e^{-|p_{1n}|w_0} + \\
&\qquad\qquad + \sum_{k=0}^{s_{n+1}-1} \frac{\Gamma(s_{n+1})}{\Gamma(s_{n+1}-k)} w_0^{s_{n+1}-k-1} \times \\
&\qquad\qquad \times \left(\frac{1}{(|p_1|+\cdots+|p_n|+|p_{1n}|)^{k+1}} - \right. \\
&\qquad\qquad\quad \left. - \frac{1}{(|p_1|+\cdots+|p_n|-|p_{1n}|)^{k+1}}\right) e^{-(|p_1|+\cdots+|p_n|)w_0} \Bigg\}.
\end{aligned}$$

Here $|p_{1n}| \equiv \sqrt{(\vec{p}_1+\cdots+\vec{p}_n)^2}$. Note that this expression has poles at $|p_{1n}| = |p_1| + \cdots + |p_n|$, i.e. when all the n momenta $\vec{p}_1,\ldots\vec{p}_n$ are collinear.

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This complicated formula is typical for the amplitudes in conventional AdS theory [14]. In this case it is somewhat tedious, though possible, to perform the recursion from section 4.2 and evaluate multi-vertex diagrams. Even more difficult is to find an effective boundary theory and the concept of ordinary AdS/CFT correspondence and/or open-closed string duality remains obscure, at least at the level of explicit multi-vertex calculations.

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